RAAGs in knot groups (Geometry and Analysis of Discrete Groups and Hyperbolic Spaces)

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B66 (2017), 037–056

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By

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Abstract

For each non-trivial knot K in S^3, we give a complete classification of right-angled Artin groups which admit embeddings into the knot group of K. We also calculate the generalized torsion elements in finite index subgroups of torus knot groups.

§ 1. Introduction and statement of results

Let Γ be a finite simplicial graph, $V(\Gamma) = \{v_1, v_2, \cdots, v_n\}$ the vertex set of Γ and $E(\Gamma)$ the edge set of Γ. Then the *right-angled Artin group* (abbreviated *RAAG*) or the *graph group* associated to Γ is the group given by the following presentation:

$$A(\Gamma) = \langle\ v_1, v_2, \ldots, v_n \ \mid\ [v_i, v_j]\ (\forall \{v_i, v_j\} \in E(\Gamma))\ \rangle.$$

RAAGs include finite rank free and free abelian groups, and share some interesting common properties. Indeed, RAAGs are linear (so residually finite) [9, 10], torsion-free, and bi-orderable as well as residually nilpotent [5], and act freely properly co-compactly on certain CAT(0) cubed complexes (cf. [3]). RAAGs have attracted group theorists and topologists. Actually, a number of studies addressed relation between RAAGs and fundamental groups of 3-manifolds (abbreviated *3-manifold groups*). The author is particularly impressed by the following complete characterization of virtual embeddings of 3-manifold groups into RAAGs, which was established by Agol [1, Theorem 1.1], Liu [17, Theorem 1.1], Przytycki-Wise [24, Corollary 1.3], [25, Corollary 1.4] et al. and was used to solve the Virtual Haken Conjecture for 3-manifolds by Agol [1, Theorem 9.1].

Received February 12, 2016. Revised March 6, 2017.
2010 Mathematics Subject Classification(s): 57M25 (05C25 20F36 20E06).
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Theorem 1.1 ([25, Corollary 1.4]). *Suppose that M is a compact aspherical 3-manifold. Then the fundamental group $\pi_1(M)$ is virtually a subgroup of some RAAG if and only if the interior of M admits a complete Riemannian metric of non-positive curvature.*

In this paper, we consider a problem in the opposite direction. Namely, we consider embeddings of RAAGs into 3-manifold groups. As preceding studies concerning embeddings of RAAGs into 3-manifold groups, we introduce Theorems 1.2 and 1.3 below. First, Droms in 1987 proved that the RAAG $A(\Gamma)$ can be embedded into a 3-manifold group only if each connected component of the defining graph Γ is either a tree or a triangle, as a part of the following more general statement.

Theorem 1.2 ([4, Theorem 2]). *For a finite simplicial graph Γ, $A(\Gamma)$ is a 3-manifold group if and only if each connected component of Γ is either a tree or a triangle.*

Besides, regarding knot groups, the free abelian groups that can be embedded in knot groups were determined. Indeed Papakyriakopoulos in 1957 proved Dehn's lemma and the asphericity of knots, and the following theorem was obtained as a consequence of these results.

Theorem 1.3 (cf. [22, Theorem 5.4.2]). *For a knot K in S^3, let $G(K)$ denotes the knot group of K. For any non-trivial knot K, the following hold.*

(1) *The boundary torus T is π_1-injective, i.e. the induced homomorphism is an embedding $\mathbb{Z}^2 \cong \pi_1(T) \hookrightarrow G(K)$.*

(2) *$\mathbb{Z}^3$ cannot be embedded in $G(K)$.*

The reader is referred to Section 2 for the definitions of the knot group and boundary torus. By using these Theorems 1.2 and 1.3, we will give a complete classification of the RAAGs which are embeddable into a given knot group as the main result of this paper.

Before stating the main result, we describe the symbols and definitions which we will use. If a group homomorphism $\psi : G \to H$ is injective, we denoted it by $\psi : G \hookrightarrow H$. For the exterior $E(K)$ of a given knot K in S^3, we have the JSJ decomposition of $E(K)$. A pair of components $\{C_1, C_2\}$ in the JSJ decomposition of $E(K)$ is said to be *adjacent* if $C_1 \cap C_2$ is a JSJ torus (cf. Section 2). We call an adjacent pair of two Seifert pieces (cf. Section 2) in the JSJ decomposition of $E(K)$ a *Seifert-Seifert gluing*. The symbols V_m, J, J_m and St_m denote the following graphs.

V_m: the graph consisting of m vertices and no edges.

J: the connected graph consisting of two vertices and a single edge joining the two vertices.

J_m: the disjoint union of m copies of J.

St_m: the join of a single vertex and m vertices.

A simplicial graph is called a *forest* if each component is a tree.

The following is the main result of this paper, which we prove in Section 4.

Theorem 1.4. *Let Γ be a finite simplicial graph, K a non-trivial knot in S^3, $E(K)$ the exterior of K and $G(K)$ the knot group of K. Then the following hold.*

(1) *If $E(K)$ has only hyperbolic pieces, then $A(\Gamma) \hookrightarrow G(K)$ if and only if Γ is equal to $V_m \coprod J_n$ for some natural numbers m and n.*

(2) *If $E(K)$ is a Seifert fibered space, then $A(\Gamma) \hookrightarrow G(K)$ if and only if Γ is equal to either V_m or St_n for some natural numbers m and n.*

(3) *If $E(K)$ has both a Seifert piece and a hyperbolic piece, and has no Seifert-Seifert gluing, then $A(\Gamma) \hookrightarrow G(K)$ if and only if Γ is a disjoint union of St_m's.*

(4) *If $E(K)$ has a Seifert-Seifert gluing, then $A(\Gamma) \hookrightarrow G(K)$ if and only if Γ is a forest.*

In addition to Theorem 1.4, we also study the following concept. For a given group G, the *graph group index* of G is defined as follows.

$$\mathrm{GI}(G) := \min\{[G : H] \mid H \text{ is a subgroup of } G, \text{ which is embedded in some RAAG}\}$$

The if part of Theorem 1.1 says that the graph group index is finite for the fundamental group of a compact 3-manifold whose interior admits a Riemannian metric of non-positive curvature. So the following question naturally arises.

Question. For a compact 3-manifold M (whose interior admits a complete Riemannian metric of non-positive curvature), how large is $\mathrm{GI}(\pi_1(M))$?

In Section 5, we give an answer to the above natural question for the torus knot groups. To be precise, we prove the following theorem as a refinement of Theorem 1.4(2).

Theorem 1.5. *Let $G(p,q)$ be the non-trivial (p,q)-torus knot group. Then we have*

$$\mathrm{GI}(G(p,q)) = pq.$$

In particular, $G(p,q)$ contains an index pq subgroup isomorphic to $A(\mathrm{St}_m)$ for $m = (p-1)(q-1)$.

Acknowledgement. First of all, I wish to sincerely thank my supervisor Makoto Sakuma for valuable discussions and his kind, helpful support. I would like to thank the members of topology seminar in Hiroshima University: Professors Makoto Sakuma, Masakazu Teragaito, Yuya Koda and senior graduate students Mikio Furokawa, Naoki Sakata and Shunsuke Aimi for their suggestions and encouragements. I am grateful to Professor Sang-hyun Kim for giving a series of beautiful lectures on embeddability between RAAGs and recommending me to read Crisp-Wiest's paper [3] at the workshop "Branched Coverings, Degenerations, and Related Topics 2014" in Hiroshima, and to Professor Michihiko Fujii for giving me a chance to present the contents of this paper at the workshop "Geometry and Analysis of Discrete Groups and Hyperbolic Spaces" at Kyoto RIMS in June 2015, and to Professor Yuichi Kabaya for giving us a series of intensive lectures on a relation between RAAGs and the Virtual Fibering Conjecture at Hiroshima University in April 2014. I also thank the unknown referee for comments and improvements on the original manuscript.

§ 2. Preliminaries

In this section, we recall the well-known facts which we will use in this paper. First of all, we recall the topology and geometry of 3-manifolds and knot exteriors. For more terminology, see [11], [14] and [26].

A *Seifert fibered space* is a compact 3-manifold that admits a foliation by circles, and can be regarded as a circle bundle over a 2-dimensional orbifold with finite exceptional points. A compact 3-manifold M is called a *hyperbolic 3-manifold* if $\pi_1(M)$ acts freely and properly discontinuously on a hyperbolic 3-space $\mathbb{H}^3$ as isometries (in particular, $\pi_1(M)$ is embedded in $\mathrm{PSL}(2,\mathbb{C})$ as a discrete subgroup), and the quotient space by the action is the interior of M.

For a knot K in S^3, let $E(K) = S^3 \setminus \mathrm{Int}(N(K))$ denote the *exterior* of K, where $N(K)$ is a tubular neighborhood of K. We call the boundary of $E(K)$ the *boundary torus*. The fundamental group, $G(K)$, of $E(K)$ is called the *knot group* of K. In this paper, we consider only non-trivial knots.

By Jaco, Shalen [12] and Johannson [13], $E(K)$ has a canonical decomposition by a family $\mathcal{T}$ of mutually disjoint tori, such that the closure C of each component of $E(K) \setminus \mathcal{T}$ satisfies one of the following conditions (1) and (2).

(1) C is a *Seifert piece*, namely, C is a Seifert fibered space.

(2) C is a *hyperbolic piece*, namely, C is a hyperbolic 3-manifold (by the Hyperbolization Theorem, cf. [14]).

This decomposition is called the *Jaco-Shalen-Johannson decomposition* (*JSJ decomposition* in brief) of $E(K)$, and each torus in $\mathcal{T}$ is called a *JSJ torus*. Note that the

JSJ decomposition gives $G(K)$ a decomposition in the sense of graph of groups, whose underlying graph is a tree. We will use this fact in the proof of Theorem 1.4. As shown in [11, IX.22. Lemma], a Seifert piece in the JSJ decomposition of $E(K)$ is one of the following (1), (2) and (3).

(1) The exterior of the (p, q)-torus knot in S^3 for some coprime numbers (p, q), namely, a Seifert fibered space, fibred over a disc with two exceptional fibers whose angles of rotations are $2\pi q/p$ and $2\pi p/q$. We denote the (p, q)-torus knot group by $G(p, q)$.

(2) A *cable space*, namely, a manifold obtained from a solid torus $S^1 \times D^2$ by removing the interior of a tubular neighborhood in $S^1 \times \mathrm{Int}D^2$ of a simple closed curve k that lies in a torus $S^1 \times S$, where S is a simple closed curve in the open disc $\mathrm{Int}D^2$ and k is non-contractible in $S^1 \times S$. The cable space is a Seifert fibered space, fibred over an annulus with a single exceptional fiber.

(3) A *composing space*, namely, a manifold homeomorphic to a product of a circle and an n holed disc for $n \geq 2$. The composing space is a Seifert fibered space, fibered over the n holed disc consisting of only regular fibres.

For a given group G and an element of $g \in G$, the centralizer of g in G is denoted by $Z_G(g)$. Then we have the following characterization of centralizers in knot groups:

Theorem 2.1 ([6, Theorem 1.1]). *Let g be a non-trivial element of a knot group $G(K)$. If $Z_{G(K)}(g)$ is not cyclic, then one of the following holds:*

(1) *There exist a JSJ torus or a boundary torus T and $h \in G(K)$ such that $g \in h\pi_1(T)h^{-1}$ and $Z_{G(K)}(g) = h\pi_1(T)h^{-1}$.*

(2) *There exist a Seifert piece M and $h \in G(K)$ such that $g \in h\pi_1(M)h^{-1}$ and $Z_{G(K)}(g) = hZ_{\pi_1(M)}(h^{-1}gh)h^{-1}$.*

Hence, up to conjugacy, every non-cyclic centralizer of an element of a knot group is either isomorphic to $\mathbb{Z}^2$ or embedded in the fundamental group of a Seifert piece. A subgroup H of a group G is said to be *malnormal* in G, if for any $g \in G \setminus H$, $gHg^{-1} \cap H = 1$.

The following theorem can be found in [8, Theorem 3].

Theorem 2.2. *Let M be a 3-manifold which is compact, connected, orientable, and irreducible. Assume moreover that the boundary ∂M has at least one component, say T, which is a torus; and that M is neither a solid torus nor a thickened torus (i.e., a product of the 2-torus and the unit interval). Denote by C the piece of the JSJ decomposition of M which contains T. Then $\pi_1(T)$ is not malnormal in $\pi_1(M)$ if and only if C is a Seifert piece.*

Finally, we introduce two key facts concerning embeddings between RAAGs.

The first one is the following lemma. Let $\mathcal{K}$ be a finite simplicial complex. A subcomplex $\mathcal{L}$ of $\mathcal{K}$ is said to be *full* (or *induced*), if $\mathcal{L}$ contains all of the simplex σ of $\mathcal{K}$ such that $\sigma^{(0)} \subset \mathcal{L}$. When $\mathcal{K}$ is a simplicial graph, a full subcomplex $\mathcal{L}$ of $\mathcal{K}$ is called a *full subgraph.*

Lemma 2.3. *Let Γ be a finite simplicial graph. If Λ is a full subgraph of Γ, then the subgroup generated by $V(\Lambda)$ is isomorphic to $A(\Lambda)$.*

Lemma 2.3 seems to be a well-known fact, and some proofs of a weak version of Lemma 2.3 can be found in [15] and [16]. The weak version states that if Λ is a full subgraph of Γ, then $A(\Lambda) \hookrightarrow A(\Gamma)$. The author could not find a proof of Lemma 2.3 and we need the above form, so we give a simple proof in Appendix.

The second one is the following theorem due to Kim-Koberda [15, Theorem 1.8].

Theorem 2.4. *Suppose that P_4 is the path graph consisting of 4 vertices and 3 edges (see Figure 2 in Section 4). Then for every finite forest Γ, we have $A(\Gamma) \hookrightarrow A(P_4)$.*

To make this paper self-contained, we give a simple alternative proof of Theorem 2.4 in Appendix. The reader is referred to [15, Theorem 1.3, Proposition 5.2] for the original proof using the "extension graph" of P_4.

§ 3. Two technical lemmas on free products with amalgamation

In this section, we prepare two technical lemmas which we will use in the proof of Theorem 1.4.

Lemma 3.1. *Let $\psi : G_1 \ast_A G_2 \to H_1 \ast_B H_2$ be a homomorphism between free products with amalgamation. If ψ satisfies the following conditions for $i = 1, 2$, then ψ is an embedding.*

(i) *$\psi(G_i) \subset H_i$ and $\psi|_{G_i} : G_i \to H_i$ is an embedding,*

(ii) *$(\psi|_{G_i})^{-1}(B) \subset A$.*

Proof. The conditions (i) and (ii) guarantee that the homomorphism ψ preserves normal forms. Thus we obtain the desired result result by the normal form theorem for free products with amalgamation (cf. [18]). □

The following lemma seems to be a standard fact. However, the author could not find a proof of the lemma in the literature. Therefore we give a proof.

Lemma 3.2. *Suppose that groups A and B share a common subgroup U such that U is malnormal in B and g is a non-trivial element of A. Then we have $Z_{A \ast_U B}(g) = Z_A(g)$.*

Proof. Note that the assertion follows immediately in the case where either $A = U$ or $B = U$. Moreover, if $g \in A \setminus U$, then the assertion follows from a classical fact [20, Theorem 4.5(ii)]. Therefore, we may assume $g \in U$, $A \neq U$ and $B \neq U$.

Suppose now, on the contrary, that there is a non-trivial element $x \in (A \ast_U B) \setminus A$ such that $gxg^{-1}x^{-1} = 1$. Let $x_1 \cdots x_n$ be a normal form of x. Since $x \notin U$, we have $n \geq 1$ and x_i belongs to $A \setminus U$ or $B \setminus U$ alternatively.

Suppose $n = 1$. Then $x = x_1$ and hence $x_1 \in B \setminus U$. Then the malnormality of U in B implies $x_1 g^{-1} x_1^{-1} \notin U$, and so $g x_1 g^{-1} x_1^{-1} \neq 1$, a contradiction.

Suppose $n \geq 2$. Note that $g \in U$ implies either $gx_1 \in A \setminus U$ or $gx_1 \in B \setminus U$. Then, observe that precisely one of the following holds.

(1) $gx_1 \in A \setminus U$, $x_1 \in A \setminus U$ and $x_2 \in B \setminus U$.

(2) $gx_1 \in B \setminus U$, $x_1 \in B \setminus U$ and $x_2 \in A \setminus U$.

In addition, we have the following claim which will be used to obtain a normal form of $gxg^{-1}x^{-1}$.

Claim 3.3. *Precisely one of the following holds.*

(1) *$x_n g^{-1} x_n^{-1} \in U$, $x_n \in A \setminus U$, $x_{n-1} \in B \setminus U$ and $x_{n-1} x_n g^{-1} x_n^{-1} x_{n-1}^{-1} \in B \setminus U$.*

(2) *$x_n g^{-1} x_n^{-1} \in A \setminus U$, $x_n \in A \setminus U$ and $x_{n-1} \in B \setminus U$.*

(3) *$x_n g^{-1} x_n^{-1} \in B \setminus U$, $x_n \in B \setminus U$ and $x_{n-1} \in A \setminus U$.*

Proof of Claim 3.3. Since $g \in U$, we see that precisely one of $x_n g^{-1} x_n^{-1} \in U$, $x_n g^{-1} x_n^{-1} \in A \setminus U$ and $x_n g^{-1} x_n^{-1} \in B \setminus U$ holds. It is easy to check that the assertions hold in each case, where we use the malnormality of U in B in the proof of (1). $\square$

To deduce a contradiction, we shall divide the proof into two cases.

Case 1. Suppose either (a) $n \geq 3$, or (b) $n = 2$ and $x_2 g^{-1} x_2^{-1} \notin U$. Apply the following operations (i), (ii) and (iii) to the word $gx_1 \cdots x_n g^{-1} x_n^{-1} \cdots x_1^{-1}$.

(i) If $gx_1 \in A \setminus U$ (resp. $gx_1 \in B \setminus U$), then regard the subword gx_1 as a single element of $A \setminus U$ (resp. $B \setminus U$).

(ii) If $x_n g^{-1} x_n^{-1} \in U$, then regard the subword $x_{n-1} x_n g^{-1} x_n^{-1} x_{n-1}^{-1}$ as a single element of $B \setminus U$.

(iii) If $x_n g^{-1} x_n^{-1} \in A \setminus U$ (resp. $x_n g^{-1} x_n^{-1} \in B \setminus U$), then regard the subword $x_n g^{-1} x_n^{-1}$ as a single element of $A \setminus U$ (resp. $B \setminus U$).

Then Claim 3.3 together with the assumption (a) or (b) guarantees that the above operations makes sense and that the resulting word is a normal form of $gxg^{-1}x^{-1}$, of length at least $2n-3$ or 3 according to whether the assumption (a) or (b) is satisfied.

Case 2. Suppose $n = 2$ and $x_2 g^{-1} x_2^{-1} \in U$. By Claim 3.3, we have

$$x_1 x_2 g^{-1} x_2^{-1} x_1^{-1} \in B \setminus U.$$

Hence,

$$(g x_1 x_2 g^{-1} x_2^{-1} x_1^{-1}) \in B \setminus U \quad (g \in U)$$

is a normal form.

Thus in each case we obtain a normal form of $gxg^{-1}x^{-1}$, and so $gxg^{-1}x^{-1} \neq 1$, a contradiction. □

§ 4. Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. We first note the following immediate consequence of Theorems 1.2 and 1.3.

Lemma 4.1. *For a finite simplicial graph Γ, if $A(\Gamma)$ is embedded in a knot group, then Γ is a forest.*

Proof. Suppose that $A(\Gamma)$ is embedded in a knot group. Then $A(\Gamma)$ is a 3-manifold group, and therefore each component of Γ is a triangle or a tree by Theorem 1.2. But, Γ cannot contain a triangle component by Theorem 1.3. Hence Γ is a forest. □

To prove Theorem 1.4(1), we study parabolic elements in $\mathrm{PSL}(2, \mathbb{C})$. Here an element g in $\mathrm{PSL}(2, \mathbb{C})$ is said to be *parabolic* if the linear fractional transformation g has a unique fixed point in $\hat{\mathbb{C}}$. For this subject, see e.g. [2], [19]. The following lemma will be used in our proof of Theorem 1.4(1).

Lemma 4.2. *If G is a non-elementary Kleinian group which has a rank 2 parabolic subgroup, then $A(J_2) \hookrightarrow G$.*

Proof. We may assume ∞ is a rank 2 parabolic fixed point of G. Then the stabilizer, P_∞, of ∞ in G is given by $P_\infty = \langle T(\omega_1), T(\omega_2) \rangle$, where $T(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $\{\omega_1, \omega_2\}$ is a pair of $\mathbb{R}$-linearly independent complex numbers. Pick a loxodromic element, g, of G. We may assume $g(\infty) = 0$. Then $P_0 := gP_\infty g^{-1}$ is given by $P_0 = \langle U(\omega_3), U(\omega_4)\rangle$, where $U(w) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$ and $\{\omega_3, \omega_4\}$ is a pair of $\mathbb{R}$-linearly independent complex numbers. We can find a finite index subgroups Q_∞ and Q_0 of P_∞ and P_0, respectively, such that the following conditions are satisfied.

(i) $Q_\infty \setminus \{1\}$ is contained in the set $\{T(z) \mid z \in \mathbb{C},\ |z| > 2\}$,

(ii) $Q_0 \setminus \{1\}$ is contained in the set $\{U(w) \mid w \in \mathbb{C},\ |w| > 2\}$.

Put $X_1 = \{z \in \hat{\mathbb{C}} \mid |z| > 1\}$ and $X_2 = \{z \in \hat{\mathbb{C}} \mid |z| < 1\}$. Then it follows that $X_1 \cap X_2 = \emptyset$, $(Q_\infty \setminus \{1\})(X_2) \subset X_1$ and $(Q_0 \setminus \{1\})(X_1) \subset X_2$. Hence the ping-pong lemma (cf. [7, II.B.]) implies $Q_\infty * Q_0 \cong \langle Q_\infty, Q_0\rangle$. Thus we obtain an embedding of $Q_\infty * Q_0 \cong A(J_2)$ into G. □

Lemma 4.3. *Let m and n be natural numbers. If Γ is a disjoint union of V_m and J_n, then $A(\Gamma) \hookrightarrow A(J_2)$.*

Proof. Let S_{J_2} be the Salvetti complex of $A(J_2)$ (cf. [3, p. 446]). Then S_{J_2} is a one point union of two 2-dimensional tori, and $\pi_1(S_{J_2}) \cong A(J_2)$. By taking an $(m+n-1)$-fold cyclic covering of S_{J_2}, we obtain an embedding of $A(J_{m+n})$ into $A(J_2)$. Now let Γ be the disjoint union of V_m and J_n. Then Γ is a full subgraph of J_{m+n}, and so $A(\Gamma) \hookrightarrow A(J_{m+n})$ by Lemma 2.3. Thus we have $A(\Gamma) \hookrightarrow A(J_2)$. □

Proof of Theorem 1.4(1). Suppose that $E(K)$ has only hyperbolic pieces.

To prove the if part, we first pick a hyperbolic piece C of $E(K)$. Then $\pi_1(C)$ is a non-elementary Kleinian group, and $\pi_1(C)$ contains the fundamental group of a boundary torus as a rank 2 parabolic subgroup. Thus Lemmas 4.2 and 4.3 imply $A(V_m \coprod J_n) \hookrightarrow \pi_1(C) \hookrightarrow G(K)$.

Conversely, suppose $A(\Gamma) \hookrightarrow G(K)$. Since the centralizer of an element of $G(K)$ is either cyclic or isomorphic to $\mathbb{Z}^2$ by Theorem 2.1, each pair of distinct edges of Γ must be disjoint (if not, then a centralizer contains either $\mathbb{Z}^3$ or F_2, a contradiction), as required. □

Let $Z(G)$ denote the center of a given group G.

Proof of Theorem 1.4(2). Suppose that $E(K)$ is a Seifert fibered space. Then K is a non-trivial (p,q)-torus knot, and so the knot group is the (p,q)-torus knot group $G(p,q)$.

We first prove the if part. Note that for any $m \geq 1$ and $n \geq 2$, we have $F_m \hookrightarrow F_n$ and $A(\mathrm{St}_m) \cong \mathbb{Z} \times F_m \hookrightarrow \mathbb{Z} \times F_n \cong A(\mathrm{St}_n)$. Hence, it is enough to show $A(\mathrm{St}_n) \hookrightarrow G(p,q)$ for some $n \geq 2$. Since K is a fibered knot, $[G(p,q), G(p,q)] \cong F_{2g}$, where

$g = (p-1)(q-1)/2 \geq 1$. Thus the subgroup of $G(p,q)$ generated by the infinite cyclic center $Z(G(p,q))$ and the free commutator subgroup splits as a direct product $Z(G(p,q)) \times [G(p,q), G(p,q)]$, which is isomorphic to $\mathbb{Z} \times F_{2g} \cong A(\mathrm{St}_{2g})$, as required.

To prove the only if part, suppose $A(\Gamma) \hookrightarrow G(p,q)$. Now we assume $E(\Gamma) \neq \emptyset$ and deduce that Γ is equal to St_n for some natural number n. The assumption $E(\Gamma) \neq \emptyset$ implies $\mathbb{Z}^2 \hookrightarrow A(\Gamma)$, we denote this subgroup by $H(\cong \mathbb{Z}^2)$. Since $Z(G(p,q))$ is infinite cyclic, H intersects $Z(G(p,q))$ non-trivially by Theorem 1.3(2) (if not, then we have $\mathbb{Z}^3 \hookrightarrow G(p,q)$, a contradiction). Hence, $A(\Gamma)$ has a non-trivial center and splits as a non-trivial direct product. Since any non-trivial free product is centerless, Lemma 4.1 implies that Γ is a tree. Therefore any connected subgraph of Γ is a full subgraph. This together with Lemma 2.3 and the fact $Z(A(P_4)) = 1$ implies that P_4 is not a subgraph of Γ. Thus Γ is equal to St_n for some natural number n. □

To prove Theorem 1.4(3), we need the following lemma.

Lemma 4.4. *Let M be either a composing space or a cable space. Then for a finite simplicial graph Γ, $A(\Gamma) \hookrightarrow \pi_1(M)$ if and only if Γ is equal to V_m or St_m for some natural number m.*

Proof. Each cable space has a finite covering homeomorphic to a composing space, and the fundamental group of a composing space is isomorphic to $A(\mathrm{St}_m)$ for some $m \geq 2$. Hence the fundamental group of a cable space has a subgroup isomorphic to $A(\mathrm{St}_m)$ for some $m \geq 2$, and this implies the if part.

The proof of the only if part is the same as that of Theorem 1.4(2), because the fundamental groups of composing spaces and cable spaces have infinite cyclic center, and do not contain a subgroup isomorphic to $\mathbb{Z}^3$. □

Remark 4.5. Let M be a Seifert piece in the JSJ decomposition of a knot exterior. Then from the proofs of Theorem 1.4(2) and Lemma 4.4, we see that $\pi_1(M)$ has a finite index subgroup isomorphic to $A(\mathrm{St}_m)$ for some $m \geq 2$. In particular, if M is the exterior of the (p,q)-torus knot, then $\pi_1(M) = G(p,q)$ has a normal subgroup $N \cong Z(G(p,q)) \times [G(p,q), G(p,q)] \cong A(\mathrm{St}_m)$ such that $\pi_1(M)/N \cong \mathbb{Z}/(pq\mathbb{Z})$.

Proof of Theorem 1.4(3). Suppose that $E(K)$ has both a Seifert piece and a hyperbolic piece, and has no Seifert-Seifert gluing.

We first prove the if part. Let Γ be the disjoint union of $\mathrm{St}_{n_1}, \cdots, \mathrm{St}_{n_k}$, and let N be the maximum of $\{n_i \mid 1 \leq i \leq k\}$. Let $\hat{\Gamma}$ be the disjoint union of k copies of St_N. Then Lemma 2.3 implies that $A(\Gamma) \hookrightarrow A(\hat{\Gamma})$, because Γ is a full subgraph of $\hat{\Gamma}$. A similar argument as in the proof of Lemma 4.3 implies $A(\hat{\Gamma}) \hookrightarrow A(\mathrm{St}_N) * \mathbb{Z}$. Since $A(\mathrm{St}_N) \hookrightarrow A(\mathrm{St}_2)$, it is enough to show that $A(\mathrm{St}_2) * \mathbb{Z} \hookrightarrow G(K)$. Now, we pick a Seifert piece, say C_1, of $E(K)$. By the assumption, there exists a hyperbolic piece, C_2, adjacent to C_1. Let T be the JSJ torus $C_1 \cap C_2$. We label the graph $\mathrm{St}_2 \coprod V_1$ as shown in the Figure 1. Let G_1, G_2 and A be the subgroup of $A(\mathrm{St}_2 \coprod V_1)$ generated by $\{u_1, v, u_2\}$, $\{v, u_2, g\}$ and $\{v, u_2\}$, respectively. Then they have the following presentations

$$G_1 = \langle u_1, v, u_2 \mid [u_1, v], [v, u_2] \rangle \cong A(\mathrm{St}_2)$$

$$G_2 = \langle v, u_2, g \mid [v, u_2] \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$$

$$A = \langle v, u_2 \mid [v, u_2] \rangle \cong \mathbb{Z}^2.$$

Moreover $A(\mathrm{St}_2) * \mathbb{Z} \cong A(\mathrm{St}_2 \coprod V_1) \cong G_1 \underset{A}{*} G_2$. The following claim completes the proof of the if part.

Claim 4.6. $G_1 \underset{A}{*} G_2 \hookrightarrow \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$.

Proof of Claim 4.6. We define a homomorphism $\psi : G_1 \underset{A}{*} G_2 \to \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$ so that the following conditions are satisfied for $i = 1, 2$.

(i) The restriction $\psi |_{G_i} : G_i \to \pi_1(C_i)$ is an embedding.

(ii) $(\psi |_{G_i})^{-1}(\pi_1(T)) = \langle v, u_2 \rangle = A$.

Then Lemma 3.1 says that such ψ is an embedding. We now explain how to construct such ψ. By the proof of Theorem 1.4(1), there exists a subgroup of $\pi_1(C_2)$ isomorphic to $P' * \langle g' \rangle$, which we continue to denote by $P' * \langle g' \rangle$. Here, P' is a finite index subgroup of $\pi_1(T)$ (so $P' \cong \mathbb{Z}^2$), and g' is a non-trivial element of $\pi_1(C_2)$. By Remark 4.5, there exists a finite index subgroup isomorphic to $A(\mathrm{St}_m)$ of $\pi_1(C_1)$ for some $m \geq 2$, which we continue to denote by $A(\mathrm{St}_m)$. Since $Z(\pi_1(C_1)) \cap P'$ is infinite cyclic, and since $[\pi_1(C_1) : A(\mathrm{St}_m)] < \infty$, we see that $Z(\pi_1(C_1)) \cap P' \cap A(\mathrm{St}_m)$ is also infinite cyclic. Let v' be the generator of $Z(\pi_1(C_1)) \cap P' \cap A(\mathrm{St}_m)$. Since $P' \cong \mathbb{Z}^2$, there exists a non-trivial element $\tilde{u}_2$ of P' such that $\langle v', \tilde{u}_2 \rangle \cong \mathbb{Z}^2$. Let u_2' be the generator of $\langle \tilde{u_2} \rangle \cap A(\mathrm{St}_m) \leq P'$. Then it follows that $\langle v', u_2' \rangle \cong \mathbb{Z}^2$. Therefore we have $\langle v', u_2', g' \rangle \cong \langle v', u_2' \rangle * \langle g' \rangle = \mathbb{Z}^2 * \mathbb{Z}$. By setting $\psi |_{G_2} (v) = v'$, $\psi |_{G_2} (u_2) = u_2'$ and $\psi |_{G_2} (g) = g'$, we obtain a homomorphism $\psi |_{G_2} : G_2 \to \pi_1(C_2)$ which is clearly an embedding. On the other hand, we have $u_2' \in A(\mathrm{St}_m) \setminus Z(A(\mathrm{St}_m))$, and so there exists an element u_1' of $A(\mathrm{St}_m)$ such that $\langle u_1', u_2' \rangle \cong F_2$. Since $v' \in Z(\pi_1(C_1))$, we have $\langle u_1', v', u_2' \rangle \cong \langle v' \rangle \times \langle u_1', u_2' \rangle \cong \mathbb{Z} \times F_2$. Thus, by setting $\psi(u_1) = u_1'$, we obtain a homomorphism $\psi : G_1 \underset{A}{*} G_2 \to \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$ as an extension of $\psi |_{G_2}$. For a schematic picture of the homomorphism ψ, see Figure 1. Then the restriction $\psi |_{G_1}$:

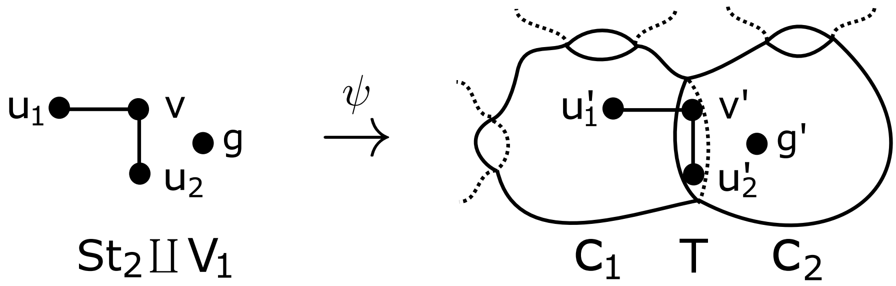

Figure 1. Labeled $\mathrm{St}_2 \coprod V_1$ and the image of the homomorphism ψ defined in the proof of Claim 4.6.

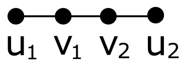

Figure 2. P_4

$G_1 \to \pi_1(C_1)$ is clearly an embedding, and so ψ satisfies the first condition (i). For $i = 1, 2$, since $\psi\,|_{G_i}$ is injective, we have the following.

$$(\psi\,|_{G_i})^{-1}(\pi_1(T)) \subset Z_{A(\mathrm{St}_2 \coprod V_1)}(u_2) \subset \langle v, u_2 \rangle.$$

Thus ψ satisfies the condition (ii), and this completes the proof of Claim 4.6. □

To prove the only if part, suppose $A(\Gamma) \hookrightarrow G(K)$. Then Γ is a forest by Lemma 4.1. So we have only to show that each component of Γ is equal to St_n for some n. Suppose this does not hold. Then Γ contains the graph P_4 as a full subgraph, and hence $A(P_4)$ is embedded in $G(K)$ by Lemma 2.3. Let u_1, v_1, v_2, u_2 be the vertices of P_4 as in Figure 2. Consider the subgroup $H = \langle u_1, v_1, v_2 \rangle$ of $A(P_4) < G(K)$. Then H is isomorphic to $A(\mathrm{St}_2)$ by Lemma 2.3. Note that the centralizer $Z_H(v_1)$ contains a subgroup isomorphic to F_2. Thus we see by Theorem 2.1 that, up to conjugacy, H is a subgroup of $\pi_1(C_1)$ for some Seifert piece C_1 of $E(K)$. On the other hand, the subgroup $A(P_4) < G(K)$ is not a subgroup of $\pi_1(C_1)$ by Theorem 1.4(2) and Lemma 4.4. Hence we have $u_2 \notin \pi_1(C_1)$.

Sublemma 4.7. *Suppose that $E(K)$ satisfies the condition of Theorem 1.4(3), C_1 is a Seifert piece, and g is a non-trivial element of $\pi_1(C_1)$. Then $Z_{G(K)}(g) = Z_{\pi_1(C_1)}(g)$.*

Proof of Sublemma 4.7. Let $\{T_i \mid 1 \le i \le n\}$ be the set of JSJ tori lie in C_1. Then we have $E(K) = (\bigcup_{i=1}^{n} B_i) \cup C_1$, where B_i is the closure of the component of $E(K) \setminus T_i$ which does not contain C_1. By our assumption and Theorem 2.2, $\pi_1(T_i)$ is malnormal in $\pi_1(B_i)$. Let $E_0 \supset E_1 \supset \cdots \supset E_n = C_1$ be the sequence of subspaces of $E(K)$ defined by

$E_0 = E(K)$ and $E_i = \overline{E_{i-1} \setminus B_i}$ $(1 \leq i \leq n)$. Note that $\pi_1(E_{i-1}) = \pi_1(E_i) \underset{\pi_1(T_i)}{*} \pi_1(B_i)$ and $Z_{G(K)}(g) = Z_{\pi_1(E_0)}(g)$.

Then, by using the malnormality of $\pi_1(T_i)$ in $\pi_1(B_i)$ and Lemma 3.2, for each $i = 1, 2, \ldots, n$, we can see that assuming $Z_{G(K)}(g) = Z_{\pi_1(E_{i-1})}(g)$ implies $Z_{G(K)}(g) = Z_{\pi_1(E_i)}(g)$. Thus we finally obtain the required result $Z_{G(K)}(g) = Z_{\pi_1(E_n)}(g) = Z_{\pi_1(C_1)}(g)$. □

Since $v_2 \in \pi_1(C_1)$, Sublemma 4.7 together with $u_2 \in Z_{\pi_1(C_1)}(v_2)$ implies $u_2 \in \pi_1(C_1)$, a contradiction. This completes the proof of Theorem 1.4(3). □

Proof of Theorem 1.4(4). Let $\{C_1, C_2\}$ be a Seifert-Seifert gluing in the JSJ decomposition of $E(K)$ and $T = C_1 \cap C_2$ the JSJ torus. The only if part is immediate from Lemma 4.1. Therefore, by Theorem 2.4, it is enough to show $A(P_4) \hookrightarrow G(K)$. We label the graph P_4 as in Figure 2. Let G_1, G_2 and A be the subgroup of $A(P_4)$ generated by $\{u_1, v_1, v_2\}$, $\{v_1, v_2, u_2\}$ and $\{v_1, v_2\}$, respectively. Then they have the following presentations

$$G_1 = \langle u_1, v_1, v_2 \mid [u_1, v_1], [v_1, v_2] \rangle \cong A(\mathrm{St}_2)$$

$$G_2 = \langle v_1, v_2, u_2 \mid [v_1, v_2], [v_2, u_2] \rangle \cong A(\mathrm{St}_2)$$

$$A = \langle v_1, v_2 \mid [v_1, v_2] \rangle \cong \mathbb{Z}^2.$$

Moreover $A(P_4) \cong G_1 \underset{A}{*} G_2$. Now, the following claim completes the proof of Theorem 1.4(4).

Claim 4.8. $G_1 \underset{A}{*} G_2 \hookrightarrow \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$.

Proof of Claim 4.8. We define a homomorphism $\psi : G_1 \underset{A}{*} G_2 \to \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$ so that the following conditions (i) and (ii) are satisfied for each $i = 1, 2$:

(i) The restrictions $\psi|_{G_i} : G_i \to \pi_1(C_i)$ is an embedding.

(ii) $(\psi|_{G_i})^{-1}(\pi_1(T)) = \langle v_1, v_2 \rangle = A$.

Then Lemma 3.1 implies that such ψ is an embedding, as required. We explain how to construct such ψ. Note that, for each $i = 1, 2$, $\pi_1(C_i)$ contains $A(\mathrm{St}_{m_i})$ for some $m_i \geq 2$ as a finite index subgroup by Remark 4.5. Then for each $i = 1, 2$, we pick a sufficiently high power, v_i', of the generator of $Z(\pi_1(C_i))$ so that v_i' lies in $A(\mathrm{St}_{m_1}) \cap A(\mathrm{St}_{m_2}) \subset \pi_1(T)$. Then we have $\langle v_1', v_2' \rangle \cong \mathbb{Z}^2$ (if not, then $\langle v_1' \rangle \cap \langle v_2' \rangle \neq 1$, and this implies that the subspace $C_1 \cup C_2$ of $E(K)$ is Seifert fibered, a contradiction). Moreover, for $j \neq i$, we pick an element u_j' of $A(\mathrm{St}_{m_j})$ so that the subgroup $\langle u_j', v_i' \rangle$ is isomorphic to F_2 (note

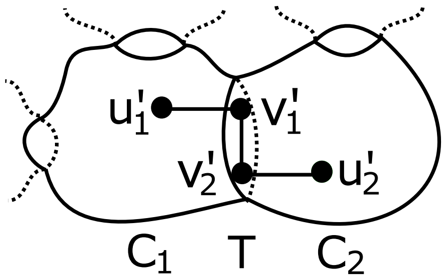

Figure 3. A schematic picture of the homomorphism ψ defined in the proof of Theorem 1.4(4).

that $v_j' \in Z(A(\mathrm{St}_{m_j})) \cong \mathbb{Z}$ implies $v_i' \in A(\mathrm{St}_{m_j}) \setminus Z(A(\mathrm{St}_{m_j}))$. By setting $\psi(u_1) = u_1'$, $\psi(v_1) = v_1'$, $\psi(v_2) = v_2'$ and $\psi(u_2) = u_2'$, we obtain a homomorphism ψ satisfying the condition (i). For a schematic picture of the image of ψ, see Figure 3. For (ii), use the fact

$$(\psi\mid_{\langle u_i, v_1, v_2\rangle})^{-1}(\pi_1(T)) \subset Z_{\langle u_i, v_1, v_2\rangle}(v_j) \subset \langle v_1, v_2\rangle \ (j \neq i,\ i = 1, 2).$$

□

This completes the proof of all assertions of Theorem 1.4. □

Remark 4.9. In [23], Niblo and Wise considered embeddings of $A(P_4)$ into compact graph manifold groups. Theorem 1.4(4) also can be obtained from Lemma 4.1, Theorem 2.4 and the result of Niblo-Wise [23, Theorem 4.2].

§ 5. Generalized torsions and experiments on torus knot groups

Let us recall the definition of generalized torsion elements in groups. An element g of a group G is said to be a *generalized torsion element* if $g \neq 1$ and there exist $g_1, g_2, \ldots, g_n \in G$ such that $g^{g_1} g^{g_2} \ldots g^{g_n} = 1$ where $g^{g_i} = g_i g {g_i}^{-1}$. A group G is said to be *generalized torsion free* if G has no generalized torsion elements.

The following lemma is essentially given in [21]. We need this lemma in the proof of Theorem 1.5.

Lemma 5.1. *Fix a presentation $\langle x, y \mid x^p = y^q\rangle$ of the (p,q)-torus knot group $G(p,q)$. If p does not divide a number i, then the element of the form $[gx^ig^{-1}, (xy)^n]$ is a generalized torsion element for any $g \in G(p,q)$ and any natural number n.*

Proof. Since $(gx^ig^{-1})^p = x^{ip}$ is an element of the center of $G(p,q)$, we have $[(gx^ig^{-1})^p, (xy)^n] = 1$. Now by using the formula $[u^n, v] = [u^{n-1}, v]^u[u, v]$ inductively, we have

$$
\begin{aligned}
1 &= [(gx^ig^{-1})^p, (xy)^n] \\
&= [(gx^ig^{-1})^{p-1}, (xy)^n]^{gx^ig^{-1}} [gx^ig^{-1}, (xy)^n] \\
&= (\ [(gx^ig^{-1})^{p-2}, (xy)^n]^{gx^ig^{-1}} [gx^ig^{-1}, (xy)^n]\ )^{gx^ig^{-1}} [gx^ig^{-1}, (xy)^n] \\
&\ \vdots \\
&= [gx^ig^{-1}, (xy)^n]^{(gx^ig^{-1})^p} [gx^ig^{-1}, (xy)^n]^{(gx^ig^{-1})^{p-1}} \cdots [gx^ig^{-1}, (xy)^n].
\end{aligned}
$$

Let us see $[gx^ig^{-1}, (xy)^n] \neq 1$. Note that the quotient group $G(p,q)/Z(G(p,q))$ is isomorphic to the free product $Q := \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z} = \langle x, y \mid x^p, y^q \rangle$. Hence, it is enough to prove that $[gx^ig^{-1}, (xy)^n] \neq 1$ in Q. Since $(xy)^n \notin g^{-1}\langle x \rangle g$, and since $Z_Q(gx^ig^{-1}) = g^{-1}Z_Q(x^i)g = g^{-1}\langle x \rangle g$, we have $[gx^ig^{-1}, (xy)^n] \neq 1$ in Q. Thus $[gx^ig^{-1}, (xy)^n] \neq 1$ in $G(p,q)$. □

Lemma 5.2. *If H is a finite index subgroup of $G(p,q)$ which is generalized torsion free, then we have*

$$pq \mid [G(p,q) : H].$$

Proof. Let W be the CW-complex associated to the presentation $G(p,q) \cong \langle x, y \mid x^p = y^q \rangle$, namely, W consists of a single vertex, two loops corresponding to the generators x and y, and a disc corresponding to the relator x^py^{-q}. We take the finite covering W_H of W corresponding to the subgroup H. Note that W_H is also a finite complex and that each edge of W_H is either labeled x or labeled y. We call an edge labeled x an x-edge, and an edge labeled y a y-edge, respectively. Since every vertex of $W_H^{(1)}$ is incident with exactly two x-edges, we see that the graph, $W_{H,x}$, obtained from $W_H^{(1)}$ by removing the y-edges, consists of disjoint simple x-cycles (see Figure 4). Thus

$$W_H^{(0)} = \coprod (\text{simple } x\text{-cycle})^{(0)}.$$

Each simple x-cycle, α_x, together with an edge-path in W_H joining α_x and the base point b of W_H, determines an element of $\pi_1(W_H, b) = H < G(p,q)$ of the form gx^ig^{-1}, where $i \geq 1$ is the girth of α_x.

Claim. *The integer p divides the integer i.*

Proof of Claim. Suppose on the contrary that p does not divide i. Note that an element of the form $(xy)^n$ belongs to H for some $n \geq 1$. Though H is generalized

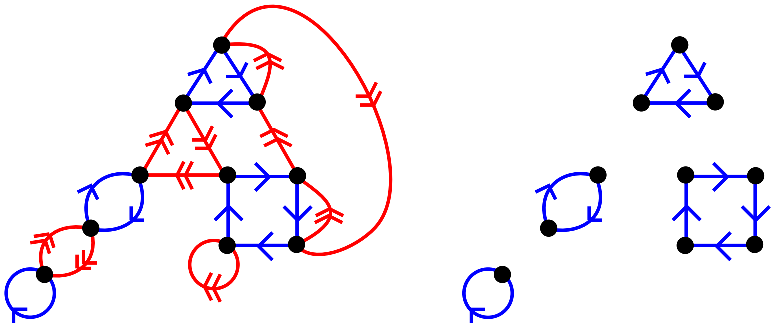

Figure 4. The left picture illustrates $W_H^{(1)}$, where each (blue) edge with a single arrow is an x-edge, and each (red) edge with a double arrow is a y-edge. The right picture illustrates the subgraph $W_{H,x}$ of $W_H^{(1)}$. Observe that $W_{H,x}$ is a disjoint union of simple x-cycles.

torsion free, Lemma 5.1 implies $[gx^ig^{-1}, (xy)^n] \neq 1$ is a generalized torsion element of H, a contradiction. $\square$

Thus p divides the girth of each simple x-cycle, and therefore p divides $\#W_H^{(0)}$. Similarly, q divides $\#W_H^{(0)}$. Since the numbers p and q are coprime, pq divides $\#W_H^{(0)} = [G(p,q) : H]$, as required. $\square$

We introduce the following concept to prove Theorem 1.5.

Definition 5.3. For a given group G, we define the *generalized torsion free index* of G as follows:

$$\mathrm{GTFI}(G) := \min\{[G : H] \mid H \leq G \text{ such that } H \text{ is generalized torsion free}\}$$

Since RAAGs are bi-orderable [5], every subgroup of a RAAG is generalized torsion free (cf. [21, Theorem 1]). This fact implies the following lemma.

Lemma 5.4. *For any group G, we have $\mathrm{GTFI}(G) \leq \mathrm{GI}(G)$.*

By Lemma 5.2, the following holds.

Lemma 5.5. *Let $G(p,q)$ be the (p,q)-torus knot group. Then we have*

$$\mathrm{GTFI}(G(p,q)) \geq pq.$$

On the other hand, Remark 4.5 implies the following lemma.

Lemma 5.6. *Let $G(p,q)$ be the (p,q)-torus knot group. Then we have*

$$\mathrm{GI}(G(p,q)) \le pq.$$

Proof of Theorem 1.5. Let $G(p,q)$ be the (p,q)-torus knot group. By using Lemmas 5.4, 5.5, and 5.6, we have

$$pq \le \mathrm{GTFI}(G(p,q)) \le \mathrm{GI}(G(p,q)) \le pq.$$

□

§ A.1. Appendix: Proofs of Lemma 2.3 and Theorem 2.4

We first prepare some symbols which we use in this Appendix.

Let Γ be a finite simplicial graph and v a vertex of Γ.

$\Gamma_{\mathrm{Lk}}(v)$: the full subgraph induced by the set of the vertices adjacent to v.

$\Gamma_{\mathrm{St}}(v)$: the full subgraph induced by $V(\Gamma_{\mathrm{Lk}(v)}) \cup \{v\}$.

$\Gamma_c(v)$: the full subgraph induced by $V(\Gamma) \setminus \{v\}$.

$D_{\Gamma_{\mathrm{St}}(v)}(\Gamma)$: the *double* of Γ along the full subgraph $\Gamma_{\mathrm{St}}(v)$, namely, $D_{\Gamma_{\mathrm{St}}(v)}(\Gamma)$ is obtained by taking two copies of Γ and gluing them along copies of $\Gamma_{\mathrm{St}}(v)$.

Lemma A.1.1. *Suppose that Γ is a finite simplicial graph and v is a vertex of Γ. Then the subgroup generated by $V(\Gamma_c(v))$ is isomorphic to $A(\Gamma_c(v))$ in $A(\Gamma)$.*

Proof. Let H be the subgroup of $A(\Gamma_c(v))$ generated by the vertices of $\Gamma_{\mathrm{Lk}}(v)$. Then $A(\Gamma)$ is the HNN extension of $A(\Gamma_c(v))$ which is presented by:

$$\langle V(\Gamma_c(v)), v \mid [v_i, v_j]\ (\forall \{v_i, v_j\} \in E(\Gamma_c(v))),\ vuv^{-1} = u\ (\forall u \in H)\ \rangle.$$

Hence $A(\Gamma_c(v))$ can be regarded as a subgroup of $A(\Gamma)$. □

Lemma 2.3 can be deduced from Lemma A.1.1 as follows.

Proof of Lemma 2.3. Suppose $V(\Gamma) \setminus V(\Lambda) = \{v_1, \ldots, v_n\}$. Then we define a descending sequence of full subgraphs, $\Gamma = \Lambda_0 \supset \Lambda_1 \supset \ldots \supset \Lambda_n = \Lambda$ by setting $\Lambda_0 = \Gamma$ and $\Lambda_i = (\Lambda_{i-1})_c(v_i)$ for $1 \le i \le n$. By successively applying Lemma A.1.1 to each pair of consecutive terms of this sequence, we obtain the desired result. □

The following easy lemma can be found in [15, Corollary 1.6 and Alternative proof of Theorem 1.3 in p. 528].

Lemma A.1.2. *Suppose that Γ is a finite simplicial graph and v is a vertex of Γ. Then $A(D_{\Gamma_{\mathrm{St}(v)}}(\Gamma))$ is embedded in $A(\Gamma)$ as an index 2 subgroup.*

Now, let us deduce Theorem 2.4 from Lemma A.1.2.

Proof of Theorem 2.4. Suppose that Γ is a finite forest. Then we can construct a finite tree T that has Γ as a full subgraph, and so by Lemma 2.3, $A(\Gamma)$ is a subgroup of $A(T)$. Thus we may assume that Γ is a tree T. Let $\mathrm{diam}(T)$ be the diameter of T, namely, the maximal length of geodesic edge-paths in T, and let $n(T)$ be the number of geodesic edge-paths in T whose length is $\mathrm{diam}(T)$. We show that $A(T) \hookrightarrow A(P_4)$ by induction on the ordered pair $(\mathrm{diam}(T), n(T))$.

Suppose $\mathrm{diam}(T) \leq 2$. Then we have $T = \mathrm{St}_m$ for some natural number m. Since St_2 is a full subgraph of P_4, and since $A(\mathrm{St}_2) \cong \mathbb{Z} \times F_2$, we have $A(T) \hookrightarrow A(\mathrm{St}_2) \hookrightarrow A(P_4)$.

Suppose $\mathrm{diam}(T) \geq 3$. Since T is a finite tree, T has a pendant vertex v (i.e., the degree of v in T is equal to 1) which is an end point of a geodesic edge-path of length $\mathrm{diam}(T)$. Let v' be the vertex adjacent to v in T, e the edge whose end points are v and v', and T' the tree obtained from T by removing v and e. Then it follows that either $\mathrm{diam}(T') < \mathrm{diam}(T)$, or $n(T') < n(T)$. To complete the proof by induction, we shall prove $A(T) \hookrightarrow A(T')$.

Case 1. suppose that the degree of v' in T' is at least 2. There are two vertices of T', v_1 and v_2, adjacent to v'. By applying Lemma A.1.2 to T' and v_1, we obtain a new tree $D_{\Gamma_{\mathrm{St}(v_1)}}(T')$ such that $A(D_{\Gamma_{\mathrm{St}(v_1)}}(T')) \hookrightarrow A(T')$ and T is a full subgraph of the tree $D_{\Gamma_{\mathrm{St}(v_1)}}(T')$ (the copy of v_2 in $D_{\Gamma_{\mathrm{St}(v_1)}}(T')$ corresponds to the vertex v of T). Thus by Lemma 2.3, we have $A(T) \hookrightarrow A(D_{\Gamma_{\mathrm{St}(v_1)}}(T')) \hookrightarrow A(T')$.

Case 2. Suppose that the degree of v' in T' is equal to 1. Let v'' be the vertex of T' adjacent to v', and e' the edge whose end points are v' and v'', and let T'' be the tree obtained from T' by removing v' and e'. If $\mathrm{diam}(T)$ is equal to 3, then, for some natural number m, T is a tree obtained by gluing St_2 and St_m along closed edges $e_1 \subset \mathrm{St}_2$ and $e_2 \subset \mathrm{St}_m$ so that the pendant vertex of e_1 and the non-pendant vertex of e_2 are identified (and the non-pendant vertex of e_1 and the pendant vertex of e_2 are identified). If $m = 2$, then $T = P_4$, and so $A(T) \hookrightarrow A(P_4)$. If $m \geq 3$, then by looking at a pendant vertex of T contained in St_m, and arguing as in Case 1, we obtain the desired result. Hence we may assume that $\mathrm{diam}(T)$ is at least 4. Then $\mathrm{diam}(T'')$ is at least 2. By applying Lemma A.1.2 to the tree T' and the vertex v', we obtain a new tree $D_{\Gamma_{\mathrm{St}(v')}}(T')$ such that $A(D_{\Gamma_{\mathrm{St}(v')}}(T')) \hookrightarrow A(T')$ and T is a full subgraph of the tree $D_{\Gamma_{\mathrm{St}(v')}}(T')$ (because $\mathrm{diam}(T'') \geq 2$). Thus we have $A(T) \hookrightarrow A(D_{\Gamma_{\mathrm{St}(v')}}(T')) \hookrightarrow A(T')$, as desired. $\square$

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